Crossover in diffusion equation: Anomalous and normal behaviors

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Ubiquitous phenomena exist in nature where, as time goes on, a crossover is observed between different diffusion regimes (e.g., anomalous diffusion at early times which becomes normal diffusion at long times, or the other way around). In order to focus on such situations we have analyzed particular relevant cases of the generalized Fokker-Planck equation $\int d\gamma' \tau(\gamma')[\partial^{\gamma'}\rho(x,t)]/\partial t^{\gamma'} = \int d\mu' d\nu' D(\mu',\nu')[\partial^{\mu'}[\rho(x,t)]^{\nu'}]/\partial x^{\mu'}$, where $\tau(\gamma')$ and $D(\mu',\nu')$ are kernels to be chosen; the choice $\tau(\gamma')=\delta(\gamma'-1)$ and $D(\mu',\nu')=\delta(\mu'-2)\delta(\nu'-1)$ recovers the normal diffusion equation. We discuss in detail the following cases: (i) a mixture of the porous medium equation, which is connected with nonextensive statistical mechanics, with the normal diffusion equation; (ii) a mixture of the fractional time derivative and normal diffusion equations; (iii) a mixture of the fractional space derivative, which is related with Lévy flights, and normal diffusion equations. In all three cases a crossover is obtained between anomalous and normal diffusions. In cases (i) and (iii), the less diffusive regime occurs for short times, while at long times the more diffusive regime emerges. The opposite occurs in case (ii). The present results could be easily extended to more complex situations (e.g., crossover between two, or even more, different anomalous regimes), and are expected to be useful in the analysis of phenomena where nonlinear and fractional diffusion equations play an important role. Such appears to be the case for isolated long-ranged interaction Hamiltonians, which along time can exhibit a crossover from a longstanding metastable anomalous state to the usual Boltzmann-Gibbs equilibrium one. Another illustration of such crossover occurs in active intracellular transport.

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I. INTRODUCTION

Anomalous diffusion has nowadays received a lot of attention. It is observed in several situations such as in CTAB micelles dissolved in salted water [1], the analysis of heartbeat histograms in a healthy individual [2], chaotic transport in laminar fluid flow of a water-glycerol mixture in a rapidly rotating annulus [3], subregion laser cooling [4], particle chaotic dynamics along the stochastic web associated with a $d=3$ Hamiltonian flow with hexagonal symmetry in a plane [5], conservative motion in a $d=2$ periodic potential [6], transport of fluid in porous media (see Ref. [7] and references therein), surface growth [7], and many other interesting physical systems.

A common way to classify anomalous diffusion is through the time dependence of the mean squared displacement, which typically satisfies $\langle(\Delta x)^2\rangle \propto t^\sigma$. If $\sigma>1$, $\sigma<1$, or $\sigma=1$, we have superdiffusion, subdiffusion or "normal" diffusion, respectively. Deviations from $\sigma=1$ may be obtained by considering generalizations of the diffusion equation

$$\frac{\partial\rho}{\partial t} = \frac{\partial^2\rho}{\partial x^2}, \tag{1}$$

where the diffusion constant D is set equal to unit and $\rho=\rho(x,t)$. One such extension is the nonlinear equation usually referred to as the porous medium equation [7]

$$\frac{\partial\rho}{\partial t} = \frac{\partial^2\rho^\nu}{\partial x^2}. \tag{2}$$

It has been applied to several physical situations such as percolation of gases through porous media ($\nu\geqslant 2$) [8], thin saturated regions in porous media ($\nu=2$) [9], a standard solid-on-solid model for surface growth ($\nu=3$), thin liquid films spreading under gravity ($\nu=4$) [10], among others [11]. The escape time, or mean first passage time, has also been studied by considering a nonlinear Fokker-Planck equation, leading eventually to a generalization of the Arrhenius law [12]. Also notice that Eq. (2) has been investigated in connection with nonextensive statistics [13].

Another example of generalization of Eq. (1) is fractional diffusion equations, which also have been used to analyze anomalous diffusion and related phenomena. In this direction, we consider a diffusion equation with time fractional derivative [14]

$$\frac{\partial^\gamma\rho}{\partial t^\gamma} = \frac{\partial^2\rho}{\partial x^2}. \tag{3}$$

Another possibility is to investigate a diffusion equation with spatial fractional derivative [14], i.e.,

$$\frac{\partial\rho}{\partial t} = \frac{\partial^\mu\rho}{\partial x^\mu}. \tag{4}$$

These equations can be related, for instance, to continuous time random walk models and generalized Langevin equations. In particular, Eq. (4) describes anomalous diffusion of the Lévy type (superdiffusion; see Ref. [15] and references therein). It has no finite second moment ($\langle(\Delta x)^2\rangle$ diverges). Let us mention at this point that this divergent feature can be avoided by using coupled space-time memories as worked out in Ref. [16]. We address here the case of uncoupled space and time memories as contained, for instance, in Eq. (4). This divergent behavior also occurs in Eq. (2) for ν sufficiently small. An unified discussion of Eq. (2) and Eq. (4) has been done in Ref. [17] by considering equation $\partial\rho/\partial t = \partial^{\mu}\rho^{\nu}/\partial x^{\mu}$.

Each one of the above equations presents one diffusive regime. In contrast, there are cases where more than one diffusion regime occurs. Examples of such situations are Hamiltonian systems with long-range interactions [18,19], particle diffusion in a quasi-two-dimensional bacterial bath [20], and enhanced diffusion in active intracellular transport [21]. Physical situations like these motivate us to investigate processes involving distinct diffusive regimes, for instance, cases which are characterized by $\langle(\Delta x)^2\rangle \propto t^{\sigma_1}$ for short time and $\langle(\Delta x)^2\rangle \propto t^{\sigma_2}$ for long time. A way to incorporate a set of diffusive regimes in a single equation without employing time dependent coefficients, is to consider a composition involving nonlinear and fractional derivative diffusion equations. In this direction, a quite general frame is to focus attention on the Fokker-Planck-like equation

$$\int_{\gamma_1}^{\gamma_2} d\gamma' \tau(\gamma') \frac{\partial^{\gamma'}\rho}{\partial t^{\gamma'}} = \int_{\mu_1}^{\mu_2} d\mu' \int_{\nu_1}^{\nu_2} d\nu' D(\mu',\nu') \frac{\partial^{\mu'}\rho^{\nu'}}{\partial x^{\mu'}}. \tag{5}$$

In the present paper, we study some representative cases of this equation by mixing terms related to normal and anomalous diffusions. In Sec. II, we analyze Eq. (1) with an extra nonlinear derivative term. In Sec. III, Eq. (1) with an additional time fractional derivative is investigated. In this same section, Eq. (1) with a space fractional derivative is also analyzed. Finally, we conclude in Sec. IV.

II. NONLINEAR FOKKER-PLANCK EQUATION

In order to mix normal (based on linear equation) and anomalous (based on nonlinear equation) diffusions, we start our study by considering, in Eq. (5), $\tau(\gamma') = \delta(\gamma'-1)$ and $D(\mu',\nu') = [D_1\delta(\nu'-1) + D_\nu\delta(\nu'-\nu)]\delta(\mu'-2)$. In this case, we have the nonlinear equation

$$\frac{\partial\rho}{\partial t} = D_1 \frac{\partial^2\rho}{\partial x^2} + D_\nu \frac{\partial^2\rho^\nu}{\partial x^2}. \tag{6}$$

Of course, we are using in Eq. (5) integration limits that comprise $\gamma'=1$, $\nu'=1$, $\nu'=\nu$, and $\mu'=2$. In the following analysis, we use the initial condition $\rho(x,0)=\delta(x)$, hence $\langle x\rangle = 0$ $(\forall t)$. Therefore, we focus on $\langle x^2\rangle$, which coincides with $\langle(\Delta x)^2\rangle$. Notice also that (i) $D_1>0$ and $D_\nu=0$ reduce Eq. (6) to the usual diffusion equation, whose solution is a Gaussian and $\langle x^2\rangle \propto t$; (ii) for $D_1=0$ and $D_\nu>0$, Eq. (6) becomes the porous medium equation and $\langle x^2\rangle \propto t^{2/(1+\nu)}$. Next, we address the behavior of $\langle x^2\rangle$ associated with the solutions of Eq. (6) for short and long times by investigating the possibilities $\nu>1$ and $\nu<1$, D_1, and νD_ν being positive quantities.

Before considering a general analysis of Eq. (6) based on numerical calculation, we perform an approximate analytical investigation. In order to identify the regimes exhibited by the solution of Eq. (6), we rewrite this equation as

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}\left\{[D_1 + \nu D_\nu \rho^{\nu-1}]\frac{\partial\rho}{\partial x}\right\}. \tag{7}$$

Thus, if D_1 is sufficiently larger than $\nu D_\nu \rho^{\nu-1}$, Eq. (7) leads, with good precision, to a diffusion like that corresponding to Eq. (1). For instance, the appropriate solution of Eq. (6) with $\nu<1$ and subject to the initial condition $\rho(x,0)=\delta(x)$ is, for time short enough, the Gaussian

$$\rho(x,t) = \frac{e^{-x^2/(4D_1 t)}}{(4\pi D_1 t)^{1/2}}. \tag{8}$$

By short enough time we mean all times such that $D_1 \gg \nu D_\nu [\rho(0,t)]^{\nu-1}$. However, as t increases, $\rho(0,t)$ decreases and this inequality becomes gradually reversed.

If $D_1 \ll \nu D_\nu \rho^{\nu-1}$, Eq. (7) reduces to the porous medium equation as presented in Eq. (2). Therefore, for long time, the solution of Eq. (6) with $\nu<1$ may be approximated by the q Gaussian

$$\rho(x,t) = \exp_q[-\beta(t)x^2]/Z(t), \tag{9}$$

where

$$q = 2-\nu. \tag{10}$$

Here, $\exp_q(x) \equiv [1+(1-q)x]^{1/(1-q)}$ if $1+(1-q)x \geq 0$ and $\exp_q(x)\equiv 0$ otherwise. This is the q-exponential function that naturally emerges in nonextensive statistical mechanics [22,23]. Note that Eq. (9) reduces to a Gaussian in the limit $q\to 1$ and has a long (short) tail behavior for $q>1$ $(q<1)$. Moreover, $Z(t)$ and $\beta(t)$ are given [13] by

$$\frac{Z(t)}{Z(0)}\left(\frac{\beta(t)}{\beta(0)}\right)^{1/2} = 1 \tag{11}$$

and $\beta(t) = \{2(3-q)[Z(0)(\beta(0))^{1/2}]^{q-1}\nu D_\nu t\}^{-2/(3-q)}$.

In general, for short time, the Gaussian (q Gaussian) is the solution of Eq. (6) to be employed when $q>1$ $(q<1)$; for long time, the q Gaussian (Gaussian) is the solution to be used when $q>1$ $(q<1)$.

In this work, we are mainly interested on the mean square displacement whenever it is finite. Thus, from Eq. (6) and $\langle x^2\rangle \equiv \int_{-\infty}^{\infty} dx x^2 \rho$, we verify that

$$\langle x^2\rangle = 2D_1 t + 2D_\nu \int_0^t d\bar{t} \int_{-\infty}^{\infty} d\bar{x} [\rho(\bar{x},\bar{t})]^\nu, \tag{12}$$

where we have employed the normalization condition, $\int_{-\infty}^{\infty} dx \rho = 1$, have exchanged the integral and derivative ordering, as well as used $\lim_{x\to\pm\infty} x^2 \partial\rho^\delta/\partial x = 0$ and $\lim_{x\to\pm\infty} x\rho^\delta/\partial x = 0$ ($\delta=1$, ν). Therefore, for $\nu<1$, we obtain

$$\langle x^2\rangle(t)\approx\begin{cases}2D_1t & \text{for small } t\\ \dfrac{1}{5-3q}\{2(3-q)[Z(0)(\beta(0))^{1/2}]^{q-1}\nu D_\nu t\}^{2/(3-q)} & \text{for large } t,\end{cases} \tag{13}$$

where the Gaussian (8) and the q Gaussian (9) are respectively employed in Eq. (12) for the cases with small and large t. For $\nu>1$, the calculation is similar to that just discussed for $\nu<1$. In this case, as time increases, we first observe anomalous (sub) diffusion and later on normal one.

If necessary, the previous calculation can be improved by considering corrections for Eq. (8). In this direction, we can express Eq. (6), with the initial condition $\rho(x,0)=\delta(x)$, in an integral form

$$\rho(x,t)=\frac{e^{-x^2/(4D_1t)}}{(4\pi D_1t)^{1/2}}+D_\nu\int_0^t d\overline{t}\times\int_{-\infty}^{\infty}d\overline{x}K(x-\overline{x},t-\overline{t})[\rho(\overline{x},\overline{t})]^\nu, \tag{14}$$

where

$$K(x-\overline{x},t-\overline{t})=\frac{\partial^2}{\partial\overline{x}^2}\left\{\frac{e^{-(x-\overline{x})^2/[4D_1(t-\overline{t})]}}{[4\pi D_1(t-\overline{t})]^{1/2}}\right\}. \tag{15}$$

We may recursively solve this equation in order to obtain an approximate solution for short time. For instance, by considering $\nu<1$ and the short time regime, the Fourier transform of Eq. (14) can be approximated, up to the linear contribution on D_ν, by

$$\mathcal{F}\{\rho(x,t)\}=e^{-D_1tk}-\frac{D_\nu(4\pi D_1)^{(1-\nu)/2}k^2}{\nu^{1/2}}\times e^{-D_1tk^2}\left(\frac{\nu}{(1-\nu)D_1k^2}\right)^{(1/2)(3-\nu)}\times\Gamma\left(\frac{3}{2}-\frac{\nu}{2},\frac{1-\nu}{\nu}D_1tk^2\right), \tag{16}$$

where $\mathcal{F}\{\cdots\}\equiv\int_{-\infty}^{\infty}dxe^{ikx}\cdots$ is the Fourier transform and $\Gamma(n,x)=\int_0^x dte^{-t}t^{n-1}$ is the incomplete Γ function.

A full investigation can be performed if, on top of the previous results, we implement a numerical approach. In fact, a careful numerical analysis confirms that the approximate analytical results are accurate. In particular, a simple estimation of the *crossover time* t_c may be obtained by imposing the approximate equality $\langle x^2\rangle_{D_1,D_\nu=0}\approx\langle x^2\rangle_{D_1=0,D_\nu}$. In Fig. 1, we illustrate the crossover through numerical solutions. Indeed, we verify the presence of two regimes for $\langle x^2\rangle$. In contrast with Fig. (1) where $D_1<D_\nu$, we present the situation corresponding to $D_1>D_\nu$ in Fig. 2. Note that the crossover times are transformed one into the other by interchanging the D_1 and D_ν values. Our main conclusions concerning Eq. (6) can be summarized as follows:

$$\langle x^2\rangle(t)\sim\begin{cases}t^{2/(1+\nu)} & \text{for } t\ll t_c\\ t & \text{for } t\gg t_c\end{cases} \tag{17}$$

when $\nu>1$ and

$$\langle x^2\rangle(t)\sim\begin{cases}t & \text{for } t\ll t_c\\ t^{2/(1+\nu)} & \text{for } t\gg t_c\end{cases} \tag{18}$$

when $\nu<1$ for

$$t_c\equiv[(1/(5-3q))(2(3-q)\times[Z(0)(\beta(0))^{1/2}]^{q-1}\nu D_\nu)^{2/(3-q)}]^{(\nu-1)/(1+\nu)}$$

[t_c was obtained assuming that the limit case in Eq. (13) coincides at $t=t_c$]. We emphasize that the long time $t^{2/(1+\nu)}$ result for the $\nu<1$ case can be employed only if $\nu>1/3$;

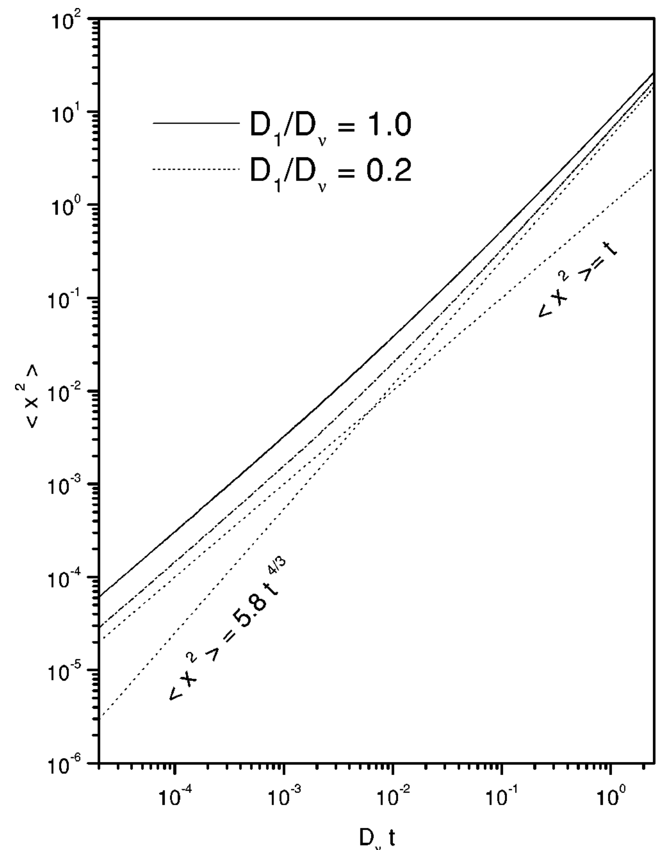

FIG. 1. Time evolution of $\langle(\Delta x)^2\rangle$ versus $\tilde{t}$, where $\tilde{t}=D_\nu t$, for the $\nu=1/2$ (superdiffusive) case, for $D_\nu=0.5$ and typical values of D_1.

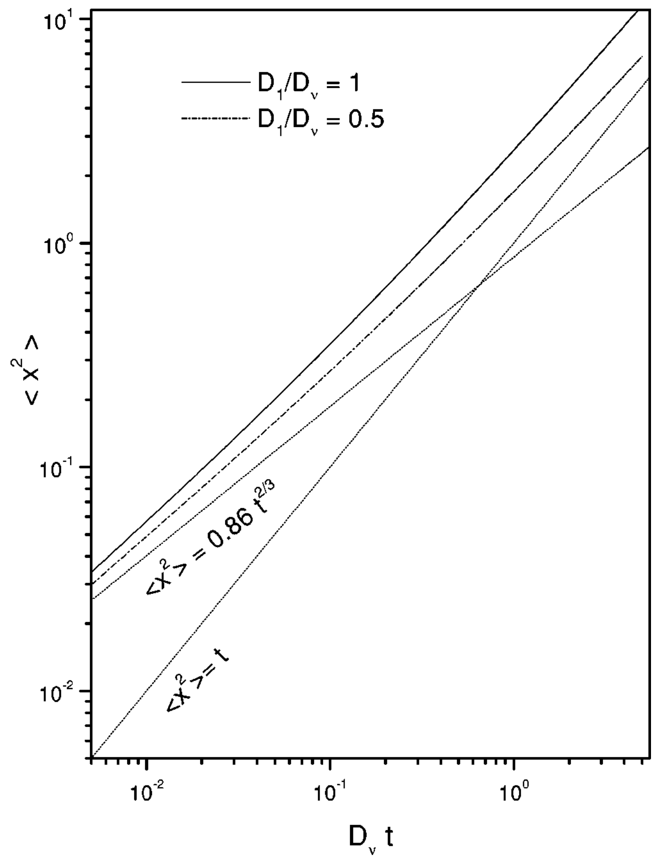

FIG. 2. Time evolution of $\langle(\Delta x)^2\rangle$ versus $\tilde{t}$, where $\tilde{t}=D_\nu t$, for the $\nu=2$ (subdiffusive) case, for $D_\nu=1$ and typical values of D_1.

indeed, $\langle x^2\rangle$ diverges for $\nu\leqslant 1/3$. For the range $\nu\leqslant 1/3$, we can employ the procedure used in the second part of Sec. III, i.e., to analyze $1/[\rho(0,t)]^2$ instead of $\langle x^2\rangle$. In this ν region, the short time behavior is governed by Gaussian regime, whereas an anomalous, Lévy-like, superdiffusive regime dominates the long time region.

III. FRACTIONAL FOKKER-PLANCK EQUATION

We will now focus on the other two particular situations of Eq. (5) where, instead of nonlinearity, we shall introduce fractional derivatives. We first address time fractional derivatives, and then space fractional derivatives. In both cases we mix normal diffusion with this type of anomalous one, and verify once again the existence of a crossover.

A. Time fractional derivative

We consider the particular case of Eq. (5), where $\tau(\gamma')=\tau_1\delta(\gamma'-1)+\tau_\gamma\delta(\gamma'-\gamma)$ and $D(\mu',\nu')=D_1\delta(\nu'-1)\delta(\mu'-2)$. Thus, the generalized Fokker-Planck equation (5) reduces to

$$\tau_1\frac{\partial\rho}{\partial t}+\tau_\gamma\frac{\partial^\gamma\rho}{\partial t^\gamma}=D_1\frac{\partial^2\rho}{\partial x^2}. \tag{19}$$

The $\gamma=2$ particular case precisely is the so called Cattaneo's equation [26], who introduced the term $\partial^2\rho/\partial t^2$ in order to take into account the finite velocity of the diffusing particles. In the more general form appearing in Eq. (19), it emerged in a discussion of continuous time random walk [27].

As in the preceding section, we are interested in the time behavior of the mean square displacement. With this aim, we use Eq. (19) as well as the assumptions immediately described below Eq. (12) and obtain

$$\tau_1\frac{d\langle x^2\rangle}{dt}+\tau_\gamma\frac{d^\gamma\langle x^2\rangle}{dt^\gamma}=2D_1. \tag{20}$$

From now on we shall explicitly refer to Caputo's fractional derivative [25], defined as follows:

$$\frac{d^\gamma f(u)}{du^\gamma}=\begin{cases} f^{(n)}(u) & \text{if}\quad \gamma=n\in N\\ \dfrac{1}{\Gamma(n-\gamma)}\displaystyle\int_0^\infty\frac{f^{(n)}(v)}{(u-v)^{\gamma+1-n}}dv & \text{if}\quad n-1<\gamma<n,\end{cases} \tag{21}$$

with $f^{(n)}(u)=d^nf(u)/du^n$.

Note that the conservation of $\int_{-\infty}^{\infty}dx\rho$ enables us to consider $\int_{-\infty}^{\infty}dx\rho=1$ for arbitrary time and consequently $(d^kdt^k)\int_{-\infty}^{\infty}dx\rho=0$ for $k\geqslant 1$. The last condition can be accomplished if the initial condition $\partial^k\rho(x,0)/\partial t^k=0$ for $k\geqslant 1$ is employed. This condition and $\rho(x,0)=\delta(x)$ lead to $\langle x^2\rangle(0)=0$ and $d^k\langle x^2\rangle(0)/dt^k=0$ for $k\geqslant 1$. We remark that our assumptions concerning the initial conditions are completely compatible with $\langle x^2\rangle\propto t^\sigma$ with $\sigma>0$. These initial conditions for $\langle x^2\rangle$ are particularly useful when solving Eq. (19) via Laplace transform, since we can use the Caputo's formula

$$\mathcal{L}\left\{\frac{d^\gamma f(u)}{du^\gamma}\right\}=s^\gamma\mathcal{L}\{f(u)\}-\sum_{k=0}^{n-1}s^{\gamma-1-k}f^{(k)}(0^+), \tag{22}$$

where $\mathcal{L}\{\cdots\}=\int_0^\infty dte^{-st}\cdots$ is the Laplace transform and $n-1<\gamma\leqslant n$. Within the previous initial conditions, the Laplace transform of Eq. (20) yields

$$\tau_1 s\mathcal{L}\{\langle x^2\rangle\}+\tau_\gamma s^\gamma\mathcal{L}\{\langle x^2\rangle\}=\frac{2D_1}{s}. \tag{23}$$

To solve Eq. (23) it is convenient to fix the γ range. We initially set $1<\gamma<2$. In this case, the mean square displacement can be written in terms of an inverse Laplace transform:

$$\langle x^2\rangle(t)=\mathcal{L}^{-1}\left\{\frac{2D_1}{\tau_\gamma s^{\gamma+1}}\frac{1}{1+\tau_1 s^{1-\gamma}/\tau_\gamma}\right\}. \tag{24}$$

To calculate this inverse Laplace transform we employ the convolution theorem with

$$\mathcal{L}\left\{\frac{1}{s^{\gamma+1}}\right\}=\frac{t^\gamma}{\Gamma(\gamma+1)} \tag{25}$$

and

$$\mathcal{L}\left\{\frac{1}{1+\tau_1 s^{1-\gamma}/\tau_\gamma}\right\}=\sum_{k=0}^{\infty}\left(-\frac{\tau_1}{\tau_\gamma}\right)^k\frac{t^{n(\gamma-1)-1}}{\Gamma[n(\gamma-1)]}. \quad (26)$$

Thus, the exact solution of Eq. (20) with $1<\gamma<2$, and subject to the initial conditions $\langle x^2\rangle(0)=0$ and $d\langle x^2\rangle(0)/dt=0$, is

$$\langle x^2\rangle(t)=\frac{2D_1}{\tau_\gamma\Gamma(\gamma+1)}\sum_{k=0}^{\infty}\left(-\frac{\tau_1}{\tau_\gamma}\right)^k\frac{1}{\Gamma[k(\gamma-1)]}\times\int_0^t d\bar{t}(t-\bar{t})^{\gamma}\bar{t}^{k(\gamma-1)-1}$$
$$=2D_1\frac{t^\gamma}{\tau_\gamma}\mathrm{E}_{\gamma-1,\gamma+1}\left(-\frac{\tau_1 t^{\gamma-1}}{\tau_\gamma}\right), \quad (27)$$

where

$$\mathrm{E}_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)} \quad (28)$$

is the generalized Mittag-Leffler function [24].

From an analysis of this function, we verify that the mean square displacement presents two characteristic regimes governed by two power laws, one for small t and the other for large t. They are given by

$$\langle x^2\rangle(t)\approx\begin{cases}[2D_1/(\tau_\gamma\Gamma(\gamma+1))]t^\gamma & \text{for small } t\\ [2D_1/\tau_1]t & \text{for large } t.\end{cases} \quad (29)$$

A similar calculation can be performed for the case $0<\gamma<1$, where we only need to specify the initial condition $\langle x^2\rangle(0)$. The result is

$$\langle x^2\rangle(t)\approx\begin{cases}[2D_1/\tau_1]t & \text{for small } t\\ [2D_1/(\tau_\gamma\Gamma(\gamma+1))]t^\gamma & \text{for large } t.\end{cases} \quad (30)$$

We pay attention onto the fact that the present results, Eq. (29) and Eq. (30), are in agreement with those obtained in Ref. [27]. Note also that the power laws (29) and (30) can, alternatively, be obtained by considering only the appropriate terms in Eq. (19). For instance, when $1<\gamma<2$, we neglect the term $\tau_\gamma\partial^\gamma\rho/\partial t^\gamma$ for short time, whereas we neglect the term $\tau_1\partial\rho/\partial t$ for long time.

B. Space fractional derivative

We consider now the fractional diffusion equation

$$\frac{\partial\rho}{\partial t}=D_1\frac{\partial^2\rho}{\partial x^2}+D_\mu\frac{\partial^\mu\rho}{\partial x^\mu}, \quad (31)$$

where D_1 and D_μ being the diffusion coefficients and $0<\mu<2$ (if $\mu=2$, the usual diffusion is recovered). This equation is obtained from Eq. (5) if we take $\tau(\gamma')=\delta(\gamma'-1)$ and $D(\mu',\nu')=\delta(\nu'-1)[D_1\delta(\mu'-2)+D_\mu\delta(\mu'-\mu)]$. For the space fractional derivative, we use the Riesz operator [14], hence, by employing the Fourier transform, we obtain,

$$\mathcal{F}\left\{\frac{\partial^\mu f(x)}{\partial x^\mu}\right\}\equiv-|k|^\mu\mathcal{F}\{f(x)\} \quad (32)$$

analysis of the times. Thus, from Eq. (31), we obtain

$$\frac{d\mathcal{F}\{\rho(x,t)\}}{dt}=-(D_1k^2+D_\mu|k|^\mu)\mathcal{F}\{\rho(x,t)\} \quad (33)$$

and consequently $\mathcal{F}\{\rho(x,t)\}=\mathcal{F}\{\rho(x,0)\}\exp(-D_2k^2t)\exp(-D_\mu|k|^\mu t)$. Furthemore, by using the convolution theorem and the initial condition $\rho(x,0)=\delta(x)$, we verify that an exact solution for Eq. (31) is given by

$$\rho(x,t)=\int_{-\infty}^{\infty}d\bar{x}\frac{e^{-(x-\bar{x})^2/(4D_1t)}}{(4\pi D_1t)^{1/2}}L_\mu(|\bar{x}|,D_\mu t), \quad (34)$$

where

$$L_\mu(|x|,t)=\int_0^\infty\frac{dk}{\pi}\cos(kx)e^{-|k|^\mu t} \quad (35)$$

is the Lévy distribution [28]. Note that Eq. (34) reduces to a Gaussian (Lévy) distribution if $D_\mu=0$ $(D_1=0)$.

To characterize the anomalous diffusion described by Eq. (31) or alternatively by Eq. (34), we cannot employ $\langle(\Delta x)^2\rangle$ since it diverges. To overcome this difficulty we employ $1/[\rho(0,t)]^2$ instead of $\langle(\Delta x)^2\rangle$. We notice that this approach to investigate Eq. (31) is close to analyzing the time dependence of $\langle(\Delta x)^2\rangle$ since in the usual case $(D_\mu=0)$ $\rho(0,t)$ essentially contains the same information displayed by $\langle x^2\rangle$. In fact, we have $\rho(0,t)\propto 1/\sqrt{\langle x^2\rangle}\propto 1/\sqrt{t}$ when $D_\mu=0$ and $\rho(x,0)=\delta(x)$. Thus, in the following discussion, we use $[1/\rho(0,t)]^2$ instead of $\langle x^2\rangle$ in order to analyze, in a unified way, both cases with finite or divergent second moment.

Before giving the analysis for $\rho(x,t)$ with arbitrary μ, we focus attention on the simplest case, the Lorentzian one $(\mu=1)$. This choice enables us to illustrate our results in a simple way since Eq. (34) can be written in terms of the error function $\mathrm{Erf}(x)$. Indeed, we have

$$\rho(0,t)=\int_{-\infty}^{\infty}d\bar{x}\frac{e^{-\bar{x}^2/(4D_1t)}}{(4\pi D_1t)^{1/2}}\frac{1}{D_\mu\pi t}\frac{1}{1+(\bar{x}/(D_\mu t)^2}$$
$$=\frac{1}{\sqrt{4\pi D_1 t}}e^{(D_\mu^2t)/(4D_1)}\left\{1-\mathrm{Erf}\left[\frac{t^{1/2}}{2}\left(\frac{D_\mu^2}{D_1}\right)^{1/2}\right]\right\}, \quad (36)$$

leading to $\rho(0,t)\sim 1/\sqrt{t}$ for short t and $\rho(0,t)\sim 1/t$ for long t. In the general case $(\mu<2)$, we employ the power series expansion for $\cos(kx)$ in Eq. (35) and perform the integrations in $\bar{x}$ and k in Eq. (34) to obtain

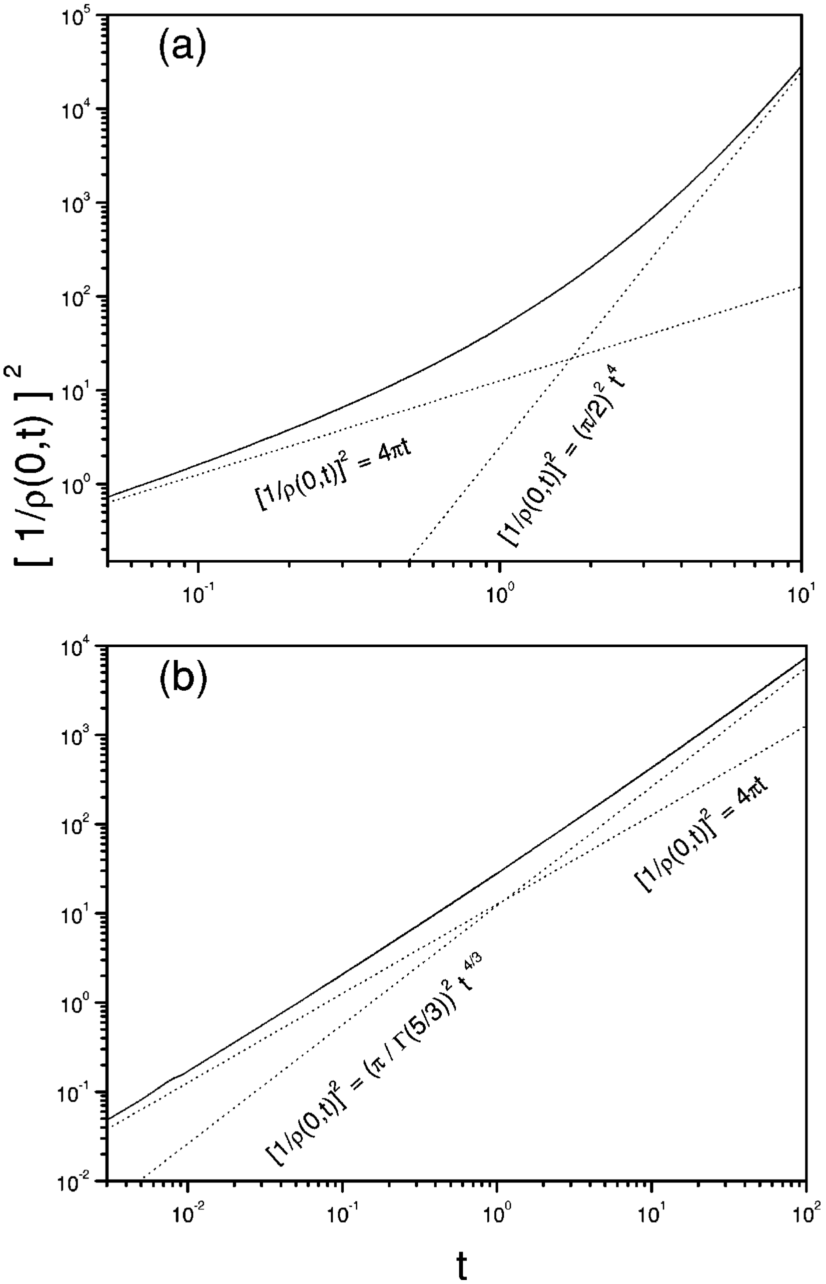

FIG. 3. Time evolution of $[1/\rho(0,t)]^2$ and of its short and long time asymptotic regimes, for typical values of μ for $D_1 = D_\mu = 1$: (a) $\mu = 1/2$, and (b) $\mu = 3/2$.

$$\rho(x,t) = \frac{1}{\mu\pi(D_\mu t)^{1/\mu}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{1}{\mu}+\frac{2}{\mu}n\right)}{\Gamma(1+n)} \left(\frac{D_1}{D_\mu^{2/\mu}}\right)^n \times t^{[1-(2/\mu)]n} {}_1F_1\left(-n,\frac{1}{2};-\frac{x^2}{4D_1 t}\right), \tag{37}$$

where ${}_1F_1(a,b;x)$ is the Kummer confluent hypergeometric function [24].

By taking the particular case $x=0$ in Eq. (37), we verify that

$$\rho(0,t) = \frac{1}{\mu\pi(D_\mu t)^{1/\mu}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{1}{\mu}+\frac{2}{\mu}n\right)}{\Gamma(1+n)} \left(\frac{D_1}{D_\mu^{2/\mu}}\right)^n \times t^{[1-(2/\mu)]n}. \tag{38}$$

A detailed analysis of this expression for small and large times leads to the result we are looking for, namely,

$$\left(\frac{1}{\rho(0,t)}\right)^2 \sim \begin{cases} t & \text{for small } t \\ t^{2/\mu} & \text{for large } t. \end{cases} \tag{39}$$

These results from Eq. (34) can be interpreted as follows. For small t, the normal diffusion is dominant, i.e., $\rho(x,t)$ is approximately governed by $\partial\rho/\partial t = D_1 \partial^2\rho/\partial x^2$ [Eq. (31) with $D_\mu = 0$]. For large t, $\rho(x,t)$ approximately obeys $\partial\rho/\partial t = D_\mu \partial^\mu \rho/\partial x^\mu$ [Eq. (31) with $D_1 = 0$], i.e., the anomalous diffusion is fully developed. Alternatively, from the solutions of these equations, results (39) are imediately recovered since

$$\rho(0,t) \sim L_{\bar{\mu}}(0, D_{\bar{\mu}} t) = \int_0^\infty \frac{dk}{\pi} e^{-|k|^{\bar{\mu}} D_{\bar{\mu}} t} = a_{\bar{\mu}} t^{-1/\bar{\mu}}, \quad (40)$$

where $a_{\bar{\mu}} = \Gamma(1 + 1/\bar{\mu})(D_{\bar{\mu}})^{-1/\bar{\mu}}/\pi$, $\bar{\mu} = 2$ for small time and $\bar{\mu} = \mu$ for large time. A set of typical crossover situations $\mu = 1/2$ and $3/2$ are illustrated in Fig. 3. Note also from Fig. 3 that an estimative of the characteristic crossover time t_c may be obtained by imposing $a_2 t_c^{-1/2} = a_\mu t_c^{-1/\mu}$, i.e., $t_c = (a_2/a_\mu)^{(2-\mu)/(2\mu)}$.

IV. SUMMARY AND CONCLUSIONS

We have analyzed diffusion equations that deviate from the usual one through the addition of extra terms, specifically either a nonlinear contribution or time (space) fractional derivatives. We focused on the mean square displacement [$\langle(\Delta x)^2\rangle$] when it is finite. On the other hand, when $\langle(\Delta x)^2\rangle$ is not finite, the diffusion field at the origin [$\rho(0,t)$] was investigated. For the nonlinear diffusion equation, which contains the usual diffusion and the porous medium one as limiting cases, two regimes were identified. One of them is related to the usual diffusion and the other to the anomalous one. The dynamics imposed by the nonlinear equation is such that the long time regime diffuses faster than the short time one. Moreover, the same conclusion is obtained when the mean square displacement is not finite.

This feature is also verified for the space fractional diffusion equation, in fact related to Lévy flights. More specifically, a normal decay is obtained for $\rho(0,t)$ for short time, and a Lévy one for long time.

This scenario is inverted in the case of time fractional derivatives. Indeed, the long time regime diffuses slower than the short time one.

Summarizing, in our investigation, $\langle(\Delta x)^2\rangle$ reduces to $\langle x^2\rangle$ and its relevant time behavior is essentially the same as that of $[1/\rho(0,t)]^2$. Indeed, the asymptotic solutions for short and long times behave as $\rho(x,t) \sim 1/\Phi(t) P(x/\Phi(t))$ and consequently $\langle x^2\rangle \propto 1/[\rho(0,t)]^2$. Notice, however, that $\rho(0,t)$ is always defined, which is not necessarily the case for $\langle x^2\rangle$. Thus, our conclusions can be put in a general scheme as follows:

$$\left(\frac{1}{\rho(0,t)}\right)^2 \sim \begin{cases} t^{\sigma_1} & \text{for small } t \\ t^{\sigma_2} & \text{for large } t, \end{cases} \quad (41)$$

where (i) $\sigma_1 < \sigma_2$ for diffusion equation including a nonlinear term [like Eq. (6)] or a space fractional derivative term [like Eq. (31)], and (ii) $\sigma_1 > \sigma_2$ for diffusion equation including a time fractional derivative term [like Eq. (19)]. We hope that the analysis presented here can be useful in the discussion of phenomena involving anomalous diffusion with two or more regimes, mainly when nonlinear and fractional diffusion equations may play an important role.

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